J. Fluid Mech. (1983), *vol.* 132, *pp.* 185–196
Printed in Great Britain

Resistance and stability of a line of particles moving near a wall

By SIMON L. GOREN
Department of Chemical Engineering, University of California, Berkeley, California 94720

(Received 2 September 1982 and in revised form 15 November 1982)

Corrections to Stokes' law are determined to first order in a/b and a/h for a sphere of radius a in a one-dimensional array of identical spheres having centre-to-centre-spacing b and translating a distance h from a no-slip wall. When h/b is small the drag is greater than that given by Stokes' law; as h/b increases, the drag generally decreases and becomes less than that given by Stokes' law. Stability of the array is examined. Motion along the line of centres is found to be stable, but the other two motions are unstable. The wall is a stabilizing influence when motion is toward the wall and a destabilizing influence when motion is away from the wall. For motion parallel to the wall, the presence of the wall shifts the region of maximum instability to smaller wavelengths. Crowley's results, which neglect any influence of the wall, are approached for h/b greater than about 5.

1. Introduction

Movement of a swarm of particles through a fluid occurs in many engineering and natural processes. The velocity of sedimentation usually decreases markedly from that of an isolated particle as the particle concentration increases. This phenomenon, known as hindered settling, is explained qualitatively by the fact that upward-moving fluid displaced by the downward-moving particles results in a larger relative velocity and larger drag force. Various theoretical models have been proposed to give the dependence of settling velocity on particle concentration. Happel & Brenner (1965) thoroughly reviewed the earlier work of Burgers, Brinkman, Happel, Kuwabara, Hasimoto and others. All of these theories require the particles to be distributed more or less uniformly throughout the fluid. An approach which first requires the determination of the probability distribution for the separation of two particles has been given by Batchelor (1972).

If the spatial distribution of particles is altered by aggregation, clustering or clumping, then the velocity of sedimentation can be significantly increased. This increase is explained qualitatively by the onset of large-scale convective motions (free convection or sedimentary flow) arising from regions of effective fluid density different from the suspension as a whole because of different local particle concentrations. Especially dramatic effects have been observed by Weiland (1982), who found that a small concentration of buoyant particles accelerated clustering and greatly enhanced the rate of settling of heavier particles.

A theory of cluster formation for initially uniform suspensions could be of considerable value. Such a theory might suggest conditions that promote clustering and more rapid settling when that is a desired goal. The theory might suggest conditions that inhibit clustering and result in deposits of controlled geometry when that is a desired goal. Although the hydrodynamic conditions are very different from low-Reynolds-number sedimentation, such a theory might provide some insight into bubble formation in fluidized beds.

The closest approach to such a theory has been developed by Crowley (1971). Crowley studied the instability of a one-dimensional periodic array of widely spaced identical particles sedimenting with velocity W in a direction normal to the line of centres through an otherwise infinite volume of fluid. Hydrodynamic interactions between the particles were taken into account to the first order in a/b, where a is the radius of the spheres and b is the centre-to-centre spacing between adjacent spheres. Crowley found the array to be unstable to infinitesimal perturbations in position and velocity. A wave consisting of six particles gives the maximum growth rate when hydrodynamic interactions between all the particles are considered. If only nearest-neighbour interactions are taken into account, then the wave of maximum growth rate consists of four particles.

Crowley did not attempt to compute the undisturbed sedimentation velocity W of the array. In fact, his approach is not capable of giving this velocity. The reason for this is that the fluid velocity induced by the motion of a particle at low Reynolds number decays as the inverse first power of distance at remote positions from that particle. Calculation of the net hydrodynamic force on a given particle arising from the fluid motions induced by the movements of N other particles requires evaluation of a sum of the form $6\pi\mu a W(3a/2b)\Sigma_{n=1}^{N} n^{-1}$ (see e.g. equation (10) of this paper). Such sums diverge as N becomes infinite. In order to compute a finite sedimentation velocity for an infinite array of particles, it is necessary to take into account inertia of the fluid in the same way that Stokes' paradox for flow past a cylinder is resolved, or, for zero-Reynolds-number flows, to take into account a bounding surface. If there is a bounding surface, fluid motions induced by the movement of the particles are reflected from the surface. The reflected flows cause a force on a given particle of opposite sign but of comparable magnitude to the direct hydrodynamic interactions. The difference in these two divergent sums as N becomes infinite approaches a finite value. We have evaluated such sums, and give results in §3 of this paper for the first-order corrections to Stokes' law for a one-dimensional array translating in the presence of a wall.

As mentioned above, Crowley did not allow for a reflecting wall (or for fluid inertia) and therefore could not compute a finite sedimentation velocity. His approach must assume that a finite undisturbed sedimentation velocity exists and that the reflected flow is unimportant in determining the stability of the array. In §4 of this paper we examine the stability of a one-dimensional array in the presence of a wall, including the effect of the reflected flows. We find that Crowley's results are approached for values of h/b greater than about 5, where h is the distance of the array from the wall and b is the particle-to-particle spacing. Crowley (1976, 1977) has treated the stability of two-dimensional arrays using the same approach.

Although the one-dimensional periodic array treated here is admittedly remote from the geometry of clustering in three-dimensional suspensions, nevertheless the results are important because they suggest that omission of the reflected flow does not lead to serious errors for the stability calculation, provided that most of the particles are sufficiently far from bounding surfaces. The same procedures can be used to address the stability of dilute three-dimensional arrays. Interparticle colloidal repulsive forces that could act to oppose clustering could be included. The goal would be to obtain estimates of the timescale for clustering and the size of the resulting clusters as functions of the particle sizes and concentrations. These would be important for estimating the overall rate of sedimentation. Inclusion of the return

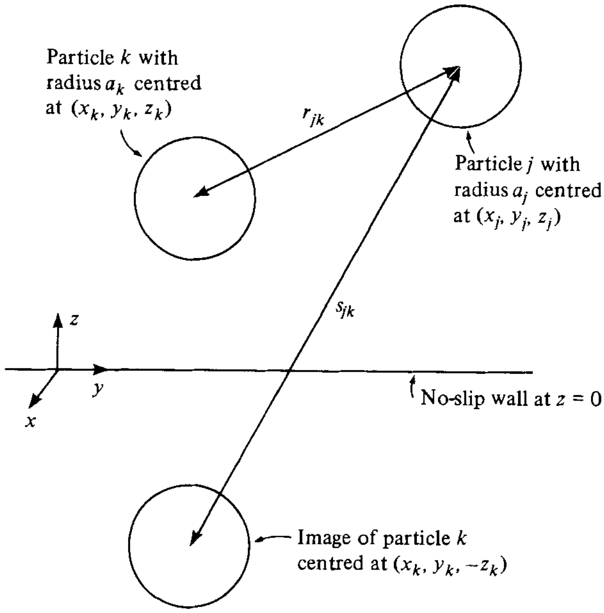

FIGURE 1. Sketch defining the geometry of spheres j and k near a wall.

flow would greatly complicate the stability calculation. It now appears unnecessary to include the return flow in the stability calculation.

2. Equations of particle motion

Consider first a single small sphere designated by the subscript k moving through a reservoir of otherwise stagnant fluid bounded by a no-slip surface at $z = 0$. Let a_k represent the radius of the sphere, (x_k, y_k, z_k) the instantaneous position of the sphere's centre, and (U_k, V_k, W_k) the instantaneous translational velocity of the sphere (see figure 1). Exact solutions to the creeping-flow equations for this geometry have been given by Brenner (1961), Maude (1961) and O'Neill (1964) through use of spherical bipolar coordinates. However, if $a_k \ll z_k$ the following relatively simple expressions can be established for the creeping flow induced at a remote point (x_j, y_j, z_j) by the motion of the sphere centred at (x_k, y_k, z_k):

$$u_{jk} = f_{jk} U_k + g_{jk}(x_j - x_k)[(x_j - x_k) U_k + (y_j - y_k) V_k + (z_j - z_k) W_k] - h_{jk} z_j (x_j - x_k) W_k, \tag{1a}$$

$$v_{jk} = f_{jk} V_k + g_{jk}(y_j - y_k)[(x_j - x_k) U_k + (y_j - y_k) V_k + (z_j - z_k) W_k] - h_{jk} z_j (y_j - y_k) W_k, \tag{1b}$$

$$w_{jk} = f_{jk} W_k + g_{jk}(z_j - z_k)[(x_j - x_k) U_k + (y_j - y_k) V_k + (z_j - z_k) W_k] + h_{jk}[(x_j - x_k) z_k U_k + (y_j - y_k) z_k V_k - (z_j^2 + z_k^2) W_k], \tag{1c}$$

where

$$f_{jk} = \frac{3a_k}{4r_{jk}} - \frac{3a_k}{4s_{jk}} - \frac{3a_k z_j z_k}{2s_{jk}^3}, \tag{2a}$$

$$g_{jk} = \frac{3a_k}{4r_{jk}^3} - \frac{3a_k}{4s_{jk}^3} + \frac{9a_k z_j z_k}{2s_{jk}^5}, \tag{2b}$$

$$h_{jk} = \frac{9a_k z_j z_k}{s_{jk}^5}, \tag{2c}$$

and
$$r_{jk} = [(x_j-x_k)^2+(y_j-y_k)^2+(z_j-z_k)^2]^{\frac{1}{2}}, \tag{3a}$$
$$s_{jk} = [(x_j-x_k)^2+(y_j-y_k)^2+(z_j+z_k)^2]^{\frac{1}{2}}. \tag{3b}$$

r_{jk} is the distance between the two points.

Use of the word 'remote' above requires $a_k \ll r_{jk}$. The terms in these equations inversely proportional to powers of r_{jk} result from application of the point force $(6\pi\mu a_k U_k, 6\pi\mu a_k V_k, 6\pi\mu a_k W_k)$ at (x_k, y_k, z_k) on the assumption that the fluid extends to infinity in all directions; these terms are in fact the limiting form of Stokes' solution when $a_k \ll r_{jk}$. The terms in the above set of equations inversely proportional to powers of s_{jk} correct the flow for the presence of the wall. These terms were derived by a mirror-image technique due to Lorentz and described by Happel & Brenner (1965). The velocity given satisfies exactly the no-slip boundary condition on the plane $z = 0$, but does not satisfy this condition on the surface of the sphere. One very important feature of the combined flow field for the present application is the fact that this flow decays with the inverse third power of distance from the sphere, whereas Stokes' flow decays only as the inverse first power of distance. To the level of accuracy considered, the motion of sphere k is resisted by a drag force with components

$$\left[-6\pi\mu a_k U_k\left(1+\frac{9a_k}{16z_k}\right), -6\pi\mu a_k V_k\left(1+\frac{9a_k}{16z_k}\right), -6\pi\mu a_k W_k\left(1+\frac{9a_k}{8z_k}\right)\right]. \tag{4}$$

This result is accurate to the first power of a_k/z_k when this ratio is small.

Now, suppose that a second small sphere of radius a_j is centred at (x_j, y_j, z_j). The fluid motion induced by sphere k exerts a drag force on sphere j given as a first approximation by $(6\pi\mu a_k u_{jk}, 6\pi\mu a_k v_{jk}, 6\pi\mu a_k w_{jk})$ with (u_{jk}, v_{jk}, w_{jk}) given by (1)–(3). If there are a number of moving spheres indicated by the running subscript k at different widely spaced positions and moving with different velocities, then the total drag force on sphere j by the induced flows is the sum of such terms. This is a consequence of the linearity of the equations of creeping motion and the level of approximation adopted for this work. Rotation of the spheres also induces a drag force on sphere j, but because fluid motion induced by rotation decays one power of r_{jk} faster than fluid motion induced by translation, this force is of order a_k/r_{jk} times that just written and is therefore neglected in view of the smallness of a_k/r_{jk}.

We suppose that sphere j is acted upon by an external force with components (F_{jx}, F_{jy}, F_{jz}). If sphere j is caused to move with velocity (U_j, V_j, W_j) in response to the external force or the induced flows of the other particles, then sphere j will also experience a drag force given by (4), but with k replaced by j. In view of the symmetry of spherical particles, for creeping-flow conditions there is no coupling between the rotation and translation of sphere j even with the presence of the wall.

In the absence of inertia, Brownian diffusion and interparticle forces, the motion of particle j in the presence of other particles and the wall is governed by the equations

$$F_{jx} = 6\pi\mu a_j U_j\left(1+\frac{9a_j}{16z_j}\right) - 6\pi\mu a_j \Sigma u_{jk}, \tag{5a}$$

$$F_{jy} = 6\pi\mu a_j V_j\left(1+\frac{9a_j}{16z_j}\right) - 6\pi\mu a_j \Sigma v_{jk}, \tag{5b}$$

$$F_{jz} = 6\pi\mu a_j W_j\left(1+\frac{9a_j}{8z_j}\right) - 6\pi\mu a_j \Sigma w_{jk}. \tag{5c}$$

The sums are to be made over all particles k except the particle j. Each of the particles satisfies a similar set of equations.

3. Unperturbed motion of a one-dimensional array

For the remainder of this paper, we take the spheres to be identical with radius a, each sphere subject to the identical external force (F_x, F_y, F_z). The spheres are *initially* in a periodic array along the x-axis with centre-to-centre spacing between adjacent spheres b, and all are the distance h from the $z = 0$ plane. The dimensionless ratios a/b and a/h are required to be much smaller than unity. Let γ represent the ratio h/b, which need not be small. If the array maintains its periodicity, then by the symmetry of the problem each sphere will have the same velocity (U, V, W). We proceed now to compute these undisturbed velocities.

We have
$$x_k = x_j + (k-j)\,b = x_j + nb \quad (n = \pm 1, \pm 2, \ldots), \tag{6a}$$
$$y_k = y_j = 0, \quad z_k = z_j = h. \tag{6b, c}$$

Substitution into (3), (2) and (1) gives
$$r_{jk} = [n^2b^2]^{\frac{1}{2}} = b|n|, \tag{7a}$$
$$s_{jk} = [n^2b^2 + 4h^2]^{\frac{1}{2}} = b(n^2+4\gamma^2)^{\frac{1}{2}}, \tag{7b}$$
$$f_{jk} = \frac{3a}{4b}[|n|^{-1} - (n^2+4\gamma^2)^{-\frac{1}{2}} - 2\gamma^2(n^2+4\gamma^2)^{-\frac{3}{2}}], \tag{8a}$$
$$g_{jk} = \frac{3a}{4b^3}[|n|^{-3} - (n^2+4\gamma^2)^{-\frac{3}{2}} + 6\gamma^2(n^2+4\gamma^2)^{-\frac{5}{2}}], \tag{8b}$$
$$h_{jk} = \frac{9a}{b^3}\gamma^2(n^2+4\gamma^2)^{-\frac{5}{2}}, \tag{8c}$$

and
$$u_{jk} = (f_{jk} + n^2b^2g_{jk})\,U + nb^2\gamma h_{jk}\,W, \quad v_{jk} = f_{jk}\,V, \tag{9a, b}$$
$$w_{jk} = -nb^2\gamma h_{jk}\,U + (f_{jk} - 2b^2\gamma^2h_{jk})\,W. \tag{9c}$$

The functions r_{jk}, s_{jk}, f_{jk}, g_{jk} and h_{jk} are seen to be even functions of n. Consequently, when (9) are substituted into (5) and the indicated sums are made over all positive and negative integers n, the contributions from W to the force F_x exactly cancel, as do the contributions from U to F_z. The results are

$$\frac{F_x}{6\pi\mu a U} = 1 + \frac{9a}{16h} - \frac{3a}{2h}\gamma \sum_{n=1}^{\infty} [2n^{-1} - (n^2+4\gamma^2)^{-\frac{1}{2}}$$
$$-2\gamma^2(n^2+4\gamma^2)^{-\frac{3}{2}} - n^2(n^2+4\gamma^2)^{-\frac{3}{2}} + 6n^2\gamma^2(n^2+4\gamma^2)^{-\frac{5}{2}}] \equiv 1 + \frac{a}{h}G_x(\gamma), \tag{10a}$$

$$\frac{F_y}{6\pi\mu a V} = 1 + \frac{9a}{16h} - \frac{3a}{2h}\gamma \sum_{n=1}^{\infty} [n^{-1} - (n^2+4\gamma^2)^{-\frac{1}{2}} - 2\gamma^2(n^2+4\gamma^2)^{-\frac{3}{2}}] \equiv 1 + \frac{a}{h}G_y(\gamma), \tag{10b}$$

$$\frac{F_z}{6\pi\mu a W} = 1 + \frac{9a}{8h} - \frac{3a}{2h}\gamma \sum_{n=1}^{\infty} [n^{-1} - (n^2+4\gamma^2)^{-\frac{1}{2}}$$
$$-2\gamma^2(n^2+4\gamma^2)^{-\frac{3}{2}} - 24\gamma^4(n^2+4\gamma^2)^{-\frac{5}{2}}] \equiv 1 + \frac{a}{h}G_z(\gamma). \tag{10c}$$

The terms in the sums G_x, G_y and G_z decrease as n^{-3}, n^{-5} and n^{-5} respectively as n becomes much greater than γ. These infinite sums therefore converge. This is a consequence of including the flows reflected from the no-slip wall. Without the reflected flows, the terms in each sum would decrease as n^{-1} and the infinite sums would diverge.

The indicated sums were evaluated numerically for a range of $\gamma = h/b$ values.

h/b	G_x	G_y	G_z
0	$\frac{9}{16}$	$\frac{9}{16}$	$\frac{9}{8}$
0.1	0.542	0.562	1.125
0.2	0.423	0.560	1.131
0.3	0.182	0.549	1.154
0.4	−0.153	0.520	1.199
0.5	−0.555	0.468	1.255
0.6	−1.010	0.392	1.308
0.7	−1.509	0.294	1.350
0.8	−2.048	0.175	1.378
0.9	−2.623	0.038	1.389
1	−3.231	−0.116	1.384
2	−10.622	−2.311	0.689
3	−19.582	−5.291	−0.791
4	−29.562	−8.781	−2.781
5	−40.300	−12.650	−5.150
6	−51.641	−16.821	−7.821
7	−63.486	−21.243	−10.743
8	−75.760	−25.880	−13.880
9	−88.410	−30.705	−17.205
10	−101.384	−35.697	−20.698

Table 1. Hydrodynamic factors needed to compute the drag on a sphere in a periodic array moving near a wall

Results are shown in table 1. As γ becomes small, G_x, G_y and G_z approach the limiting values of $\frac{9}{16}$, $\frac{9}{16}$ and $\frac{9}{8}$ respectively, given by the method of reflections for a single sphere. In this limit the distance between a sphere and the wall is much less than the distance between a sphere and the next sphere in the array. Hydrodynamic interactions with the wall predominate and the drag forces are larger than that given by Stokes' law because of the extra resistance at the wall.

As we move the array away from the wall by increasing $\gamma = h/b$, the values of G_x and G_y decrease. At sufficiently large values of γ, these values become negative so that the drag forces are less than those given by Stokes law. This is because the cooperative effect of each particle moving in the wakes of the other particles serves to decrease the drag. As might be expected, this cooperative effect is most pronounced for the array of particles translating along the line of centres, which explains why G_x becomes negative at a lower value of γ than does G_y or G_z. The values of G_z first show a slight increase as γ is increased, but then decrease for γ greater than about 0.9.

For sufficiently large $\gamma = h/b$, large negative corrections to Stokes' law would be computed using the values given in table 1 unless a/h is sufficiently small. Large corrections are inconsistent with the restrictions $a \ll b$ and $a \ll h$. These inequalities may be rewritten as $a/b \ll \gamma$ and $a/h \ll 1/\gamma$. The larger γ becomes the smaller a/h must become in order for the first reflection only to give a good approximation for the drag forces. Consequently, the present procedure is probably reliable only for small corrections to Stokes' law. To assess the level of accuracy would require retaining higher reflections (i.e. higher powers of a/b and a/h than the first).

4. Stability of a one-dimensional array

We now examine whether the one-dimensional array retains its uniform spacing in response to infinitesimal perturbations of position and velocity of each particle. Let $(\delta x_k, \delta y_k, \delta z_k)$ represent the perturbation in position of particle k relative to the unperturbed position (x_j^0, y_j^0, z_j^0) of particle j. The superscript 0 indicates evaluation at the unperturbed position. We take the perturbations to vary sinusoidally along the line of centres, there being m particles per wave. As is customary with linear stability analyses, we also take the perturbations to vary exponentially with time. Thus we write

$$x_k = x_j^0 + nb + \delta x_k = x_j^0 + nb + \xi E_n, \tag{11a}$$

$$y_k = y_j^0 + \delta y_k = y_j^0 + \eta E_n, \tag{11b}$$

$$z_k = z_j^0 + \delta z_k = z_j^0 + \zeta E_n, \tag{11c}$$

and

$$U_k = U + \delta U_k = U + \beta \xi E_n, \tag{12a}$$

$$V_k = V + \delta V_k = V + \beta \eta E_n, \tag{12b}$$

$$W_k = W + \delta W_k = W + \beta \zeta E_n, \tag{12c}$$

where

$$E_n \equiv \exp\left(\beta t + \frac{i2\pi n}{m}\right) \tag{13}$$

and $n = 0$ for particle j. β is the amplification factor whose sign and magnitude we seek to determine. If the real part of β is positive, the infinitesimal perturbation grows exponentially in time; if the real part of β is negative, the infinitesimal perturbation decays exponentially in time. The procedure for determining β is to substitute (11) and (12) into the equations of motion (5). The resulting equations are linearized, and because (U, V, W) satisfies the equations for the initial periodic array the zeroth-order terms cancel. This leaves a set of three linear homogeneous equations for the perturbation amplitude (ξ, η, ζ), which constitutes an eigenvalue problem for β.

For an array moving normal to the wall, the undisturbed velocity W varies with position as shown in §3. The perturbations introduced in (11) and (12) are about undisturbed (time- or position-dependent) positions and velocities. Consequently, this calculation examines the stability in time in a coordinate system moving with the local undisturbed velocity. Since inertial effects are completely neglected owing to the smallness of the Reynolds number, the acceleration or deceleration of the array is unimportant. However, another approximation is implicit. We have treated this problem as one that grows in time in a moving coordinate system instead of one where disturbances grow in a fixed coordinate system. It is known from other stability calculations that the two approaches give equivalent results when the amplification achieved over one wavelength is small. The condition for this is $\beta mb/W \ll 1$ or $(\beta b^2/aW)(ma/b) \ll 1$. The calculations show that $\beta b^2/aW$ is of order unity, so that the above inequality will be satisfied if $a/b \ll 1/m$. Our calculations then give the local amplification rate β for the prevailing position. To find the total amplitude A achieved as a function of position one would have to integrate the equation $\mathrm{d}\ln A/\mathrm{d}t = W \mathrm{d}\ln A/\mathrm{d}z = \beta$ using the position-dependent values of β and W.

Because of the complexity of the equations, we have treated as three separate cases the situation where the external force is in the x-direction only, the external force is in the y-direction only, and the external force is in the z-direction only. We present here only the analysis of the third case, it being illustrative of the procedure in general.

For the constant external force $(0, 0, F_z)$ on each particle, the unperturbed velocity is $(0, 0, W)$. To the first order in small quantities, the non-zero terms in ($5a$) are

$$0 = \left(1+\frac{9a}{16h}\right)\delta U_j - \Sigma\{f^0_{jk}\,\delta U_k + g^0_{jk}\,n^2b^2\delta U_k - g^0_{jk}\,nbW(\delta z_j-\delta z_k) + h^0_{jk}\,nb^2\gamma\delta W_k$$
$$- W[h^0_{jk}\,b\gamma(\delta x_j-\delta x_k) - h^0_{jk}\,nb\delta z_j$$
$$-nbh^0_{jk}(\delta z_j-\delta z_k) - 5h^0_{jk}\,\gamma n^2b(n^2+4\gamma^2)^{-1}(\delta x_j-\delta x_k)$$
$$+10h^0_{jk}\,n\gamma^2b(n^2+4\gamma^2)^{-1}(\delta z_j-\delta z_k)]\}. \tag{14}$$

Substitute $\delta U_k = \xi\beta E_n$, $\delta x_k = \xi E_n$, etc. On making the indicated sums in (14), contributions for terms involving δz_j arising from positive n exactly cancel those arising from the corresponding negative n. For the other terms, combining the corresponding terms for positive and negative n leads to the factor $\cos(2\pi n/m)$ when the coefficients are even functions of n, and to the factor $\sin(2\pi n/m)$ when the coefficients are odd functions of n. After collecting together those terms proportional to ξ and separately those terms proportional to ζ, (14) becomes

$$\xi\left[\beta A + \frac{aW}{b^2}B\right] = \mathrm{i}\zeta\left[\beta C + \frac{aW}{b^2}D\right], \tag{15}$$

where

$$A = 1 + \frac{9a}{16h} - \frac{3a}{2b}\sum_{n=1}^{\infty}[2n^{-1} - (n^2+4\gamma^2)^{-\frac{1}{2}}$$
$$-2\gamma^2(n^2+4\gamma^2)^{-\frac{3}{2}} - n^2(n^2+4\gamma^2)^{-\frac{3}{2}} + 6n^2\gamma^2(n^2+4\gamma^2)^{-\frac{5}{2}}]\cos\frac{2\pi n}{m},$$

$$B = 18\gamma^3\sum_{n=1}^{\infty}[(n^2+4\gamma^2)^{-\frac{5}{2}} - 5n^2(n^2+4\gamma^2)^{-\frac{7}{2}}]\left[1-\cos\frac{2\pi n}{m}\right],$$

$$C = 18\gamma^3\sum_{n=1}^{\infty} n(n^2+4\gamma^2)^{-\frac{5}{2}}\sin\frac{2\pi n}{m},$$

$$D = \tfrac{3}{2}\sum_{n=1}^{\infty} n[n^{-3} - (n^2+4\gamma^2)^{-\frac{3}{2}} + 18\gamma^2(n^2+4\gamma^2)^{-\frac{5}{2}} - 120\gamma^4(n^2+4\gamma^2)^{-\frac{7}{2}}]\sin\frac{2\pi n}{m}.$$

A similar treatment of ($5c$) yields

$$\zeta\left[\beta E + \frac{aW}{b^2}F\right] = \mathrm{i}\xi\left[\beta G + \frac{aW}{b^2}H\right], \tag{16}$$

where

$$E = 1 + \frac{9a}{8h} - \frac{3a}{2b}\sum_{n=1}^{\infty}[n^{-1} - (n^2+4\gamma^2)^{-\frac{1}{2}} - 2\gamma^2(n^2+4\gamma^2)^{-\frac{3}{2}} - 24\gamma^4(n^2+4\gamma^2)^{-\frac{5}{2}}]\cos\frac{2\pi n}{m},$$

$$F = -\tfrac{9}{8}\gamma^{-2} + 18\gamma^3\sum_{n=1}^{\infty}[3(n^2+4\gamma^2)^{-\frac{5}{2}} - 20\gamma^2(n^2+4\gamma^2)^{-\frac{7}{2}}]\left[1+\cos\frac{2\pi n}{m}\right],$$

$$G = -C,$$

$$H = -\tfrac{3}{2}\sum_{n=1}^{\infty} n[n^{-3} - (n^2+4\gamma^2)^{-\frac{3}{2}} - 6\gamma^2(n^2+4\gamma^2)^{-\frac{5}{2}} - 120\gamma^4(n^2+4\gamma^2)^{-\frac{7}{2}}]\sin\frac{2\pi n}{m}.$$

When ξ and ζ are eliminated from (15) and (16), we obtain a quadratic equation for β.

For the case of the external force in the z-direction only, perturbations in the y-direction do not enter into the calculation of the growth factor β. For the other two cases, perturbations in all three directions do influence the growth factor; these cases also result in quadratic equations for β, but with different collections of infinite sums to be evaluated. All of the infinite sums converge. Even when $\gamma \to \infty$ and the

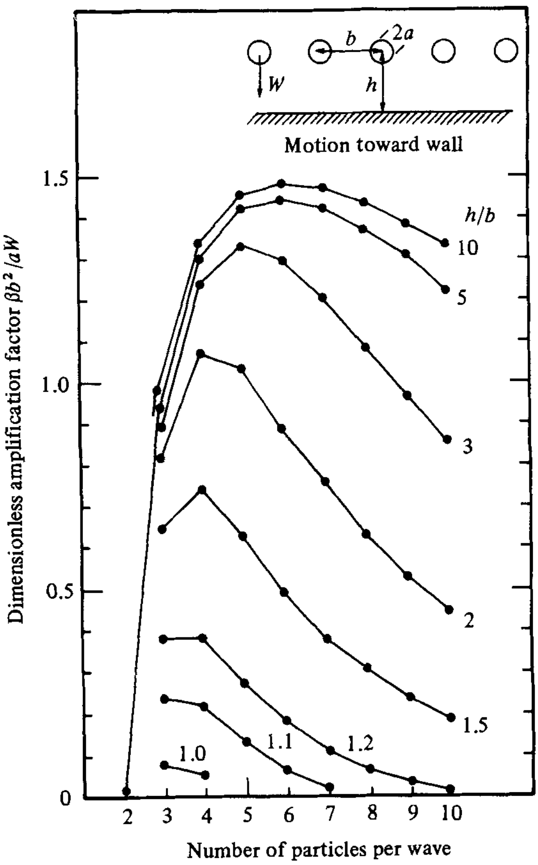

FIGURE 2. Dimensionless amplification versus number of particles per wave for motion of the array toward a wall.

array is infinitely far from the wall, i.e. when the flows reflected from the wall are neglected in computing the drag forces, the sums remain convergent. This is a consequence of the trigonometric functions in the sums.

Solution of the quadratic equation for each case determines the dimensionless growth factor $\beta b^2/aW$ for that case as a function of the two dimensionless ratios a/b and $\gamma = h/b$ and the number m of particles per wave. To the level of approximation treated here, the dependence of $\beta b^2/aW$ on a/b is neglected; this is equivalent to approximating the terms A and E above as unity when $a \ll b$ and $a \ll h$.

It is relatively straightforward to show that to this level of approximation, the limiting expression for β as γ becomes infinite is

$$\beta = \pm \frac{3aW}{2b^2} \sum_{n=1}^{\infty} n^{-2} \sin \frac{2\pi n}{m} \tag{17}$$

for both cases where the array translates normal to the line of centres. This is the result given by Crowley (1971). It is therefore clear that Crowley's procedure gives correct results for the stability of a sedimenting array far from any boundary, even though his approach is incapable of predicting the unperturbed velocity of sedimentation.

Computer programs were developed to evaluate the required sums and to solve the quadratic equations giving $\beta b^2/aW$ as functions of $\gamma = h/b$ and m.

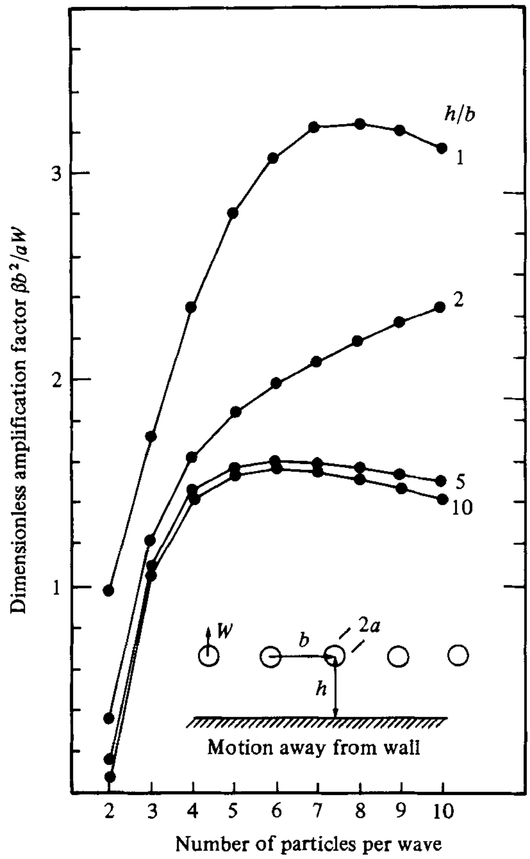

FIGURE 3. Dimensionless amplification versus number of particles per wave for motion of the array away from a wall.

When the undisturbed motion is along the line of centres, the array was found to be stable for all values of γ and for m up to 10. Morrison (1973) showed that a one-dimensional array infinitely far from boundaries and translating along its line of centres is stable.

Translation normal to the line of centres is found to be unstable. Typical results for the dimensionless amplification factor $\beta b^2/aW$ are shown in figures 2, 3 and 4 for sedimentation toward the wall, sedimentation away from the wall, and sedimentation parallel to the wall respectively.

For sedimentation toward the wall (see figure 2), the wall is seen to have a stabilizing influence on the array. As h/b decreases, the amplification factor decreases and instability is confined to a narrower region of wavelengths. No unstable waves were found for $h/b \leqslant 0.9$. The trend toward greater stability can be explained as follows. If a perturbation causes a given particle to move closer to the wall than the array as a whole, its velocity decreases relative to the velocity of the array as a whole because of the increased resistance according to (4); the faster-moving array can then catch up with the particle, thereby restoring the uniformity of the array. A similar argument holds for perturbations that cause a given particle to move further from the wall than the array as a whole. When a perturbation causes a given particle to move laterally from the array, the hydrodynamic interactions with the other particles tend to restore the lateral position of that particle.

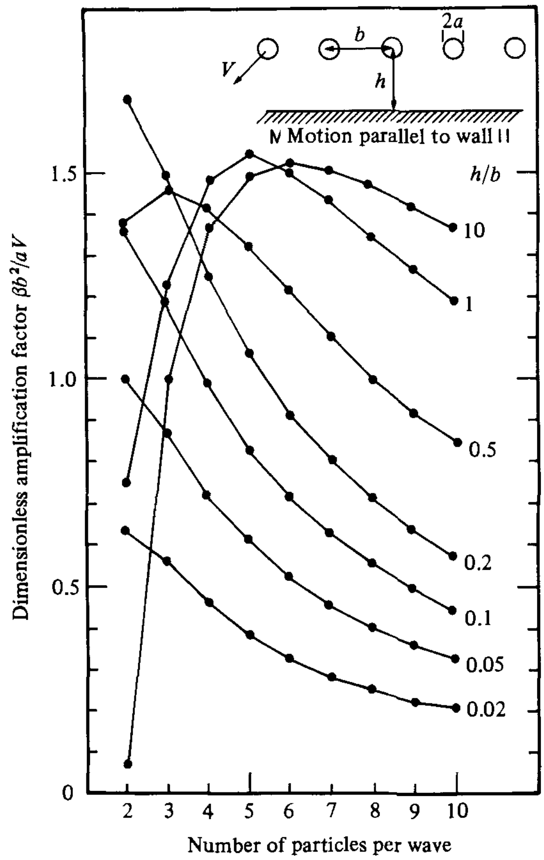

FIGURE 4. Dimensionless amplification versus number of particles per wave for motion of the array parallel to a wall.

The wall has a destabilizing influence when sedimentation is away from the wall. Figure 3 shows that, as h/b decreases, the amplification factor increases. In this case, if a perturbation causes a particle to move further from the wall than the array as a whole, its velocity increases relative to the array as a whole because of the decreased resistance according to (4); this particle will therefore continue to move further and further ahead of the array. A similar argument holds for perturbations that cause a given particle to move closer to the wall.

Figure 4 shows the complex influence of the wall on the stability of an array sedimenting parallel to the wall. The wall is stabilizing for longer-wavelength disturbances ($m \geqslant 6$) in that, as h/b decreases, the amplification factor decreases. However, for shorter-wavelength disturbances ($m < 6$) the amplification first increases as h/b decreases from a large value, reaches a maximum, and thereafter decreases as the wall is approached. The maximum amplification rates for $h/b \leqslant 0.2$ and $m = 2$ slightly exceed that for $h/b \gg 1$, which occurs at $m = 6$. The shift of the wavelength for maximum amplification to smaller values as the wall is approached can be explained as follows. Far from the wall when a perturbation causes a particle to move in a given direction, its hydrodynamic interactions with other particles tend to cause them to move in the same direction. The perturbation velocities of neighbouring particles will therefore be correlated to a certain degree. However, the flow reflected from the wall due to the motion of any given particle is to a large extent opposite

h/b	Motion toward wall $m=2$	$m=3$	$m=6$	Motion away from wall $m=2$	$m=3$	$m=6$	Motion parallel to wall $m=2$	$m=3$	$m=6$
1	-0.182	0.074	-0.063	0.972	1.723	3.092	0.749	1.235	1.500
2	< 0.001	0.815	0.899	0.376	1.226	1.973	0.375	1.082	1.553
5	< 0.001	0.943	1.446	0.150	1.093	1.601	0.150	1.026	1.530
10	< 0.001	0.978	1.485	0.075	1.053	1.560	0.075	1.018	1.524
∞	0	1.015	1.522	0	1.015	1.522	0	1.015	1.522

Table 2. Dimensionless amplification factor $\beta b^2/aW$ or $\beta b^2/aV$

in direction to the motion of that particle itself. The reflected flow consequently significantly weakens the degree of correlation in perturbation velocities, which permits (encourages) more-rapid growth of small-wavelength disturbances.

Table 2 has been prepared to illustrate the approach of the current results as h/b is increased to Crowley's results, which completely neglect the influence of the wall on the stability (i.e. $h/b = \infty$). Crowley's treatment is seen to give reasonably accurate results for h/b greater than about 5 for motion normal to the wall and for h/b greater than about 2 for motion parallel to the wall provided that m is greater than 2.

This research was supported in part by the National Science Foundation through grant CPE 8103635.

REFERENCES

Batchelor, G. K. 1972 *J. Fluid Mech.* **52**, 245.

Brenner, H. 1961 *Chem. Engng Sci.* **16**, 242.

Crowley, J. M. 1971 *J. Fluid Mech.* **45**, 151.

Crowley, J. M. 1976 *Phys. Fluids* **19**, 1296.

Crowley, J. M. 1977 *Phys. Fluids* **20**, 339.

Happel, J. & Brenner, H. 1965 *Low Reynolds Number Hydrodynamics*, chaps. 2 and 8. Prentice-Hall.

Maude, A. D. 1961 *Brit. J. Appl. Phys.* **12**, 293.

Morrison, F. A. 1973 *Chem. Engng Sci.* **28**, 1115.

O'Neill, M. E. 1964 *Mathematika* **11**, 67.

Weiland, R. 1982 *Fine Particle Society Meeting, May, Chicago.*